MTH 310 HW 7 Solutions

March 18, 2016

Section 4.2, Problem 5c

Compute $gcd(x^4 + 3x^2 + 2x + 4, x^2 - 1)$ in $\mathbb{Z}_7$.
Answer. Note that by the division algorithm, $x^4 + 3x^2 + 2x + 4 = (x^2 + 4)(x^2 - 1) + 2x + 3$ and $(x^2 - 1) = (2x + 3)(3x + 3) = 2(x + 4)(3x + 3)$, so by the Euclidean algorithm the greatest common divisor is $x + 4$ (which of course is also $x - 1$ in $\mathbb{Z}_7$). Note that the greatest common divisor is defined to be monic, so the greatest common divisor is **not** $2x + 3$.

Section 4.2, Problem 13

Let $\mathbb{F}$ be a field and $a(x), b(x), c(x) \in \mathbb{F}[x]$. If $a(x)|b(x)c(x)$ and $gcd(a(x), b(x)) = 1$, show $a(x)|c(x)$.
Answer. Since $gcd(a(x), b(x)) = 1$, there are polynomials $p(x), q(x) \in \mathbb{F}[x]$ with $1 = a(x)p(x) + b(x)q(x)$. Since $a(x)|b(x)c(x)$ there is a polynomial $d(x) \in \mathbb{F}[x]$ with $a(x)d(x) = b(x)c(x)$. Then we have that if $k(x) = (c(x)p(x) + d(x)q(x))$, then $c(x) = 1c(x) = (a(x)p(x) + b(x)q(x))c(x) = a(x)c(x)p(x) + b(x)c(x)q(x) = a(x)c(x)p(x) + a(x)d(x)q(x) = a(x)(c(x)p(x) + d(x)q(x)) = a(x)k(x)$, so $a(x)|c(x)$.

Section 4.3, Problem 6

Show that $x^2 + 1$ is irreducible in $\mathbb{Q}[x]$.
Answer. Assume not. Then $x^2 + 1 = (ax + b)(cx + d)$ for some $a, b, c, d \in \mathbb{Q}$, where $a \neq 0 \neq c$. This implies that if $y = \frac{-b}{a}$, $y^2 + 1 = 0$, which implies that there is a square root of -1 in $\mathbb{Q}$, a contradiction.